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The Cosserat spectrum for cylindrical geometries (Part 2: $\tilde{u}^{(-1)}$ subspace and applications)

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Abstract

We construct the orthonormal bases of the Cosserat subspace $\tilde{\boldsymbol{u}}^{(-1)}$ corresponding to the eigenvalue of infinite multiplicity $\tilde{\omega} = -1$ for the first boundary value problems of elasticity for a solid cylinder and a cylindrical rigid inclusion. These bases involve the Jacobi polynomials with different weight functions. An example of non-harmonic heat flow past a cylindrical rigid inclusion shows that the sequence of $\tilde{\boldsymbol{u}}^{(-1)}$ converges fast, thus, the Cosserat spectrum theory is an efficient method for solving elasticity problems of general body force or boundary loading.

1. Cosserat subspace $\tilde{\boldsymbol{u}}^{(-1)}$ in cylindrical coordinate system

The Cosserat eigenvectors are composed of the discrete subspace of eigenvectors $\tilde{\boldsymbol{u}}_n$, the subspace of eigenvectors $\tilde{\boldsymbol{u}}_n^{(-1)}$ corresponding to the eigenvalue of infinite multiplicity $\tilde{\omega} = -1$ and the subspace of eigenvectors $\tilde{\boldsymbol{u}}^{(\infty)}$ corresponding to the eigenvalue of infinite multiplicity $\tilde{\omega} = -\infty$ (Mikhlin, 1973). In Part 1, we obtained the discrete Cosserat eigenvalues $\tilde{\omega}_n$ and eigenvectors $\tilde{\boldsymbol{u}}_n$ for cylindrical bodies. Part 2 is devoted to the construction of the orthogonal bases of the Cosserat eigenvectors $\tilde{\boldsymbol{u}}_n^{(-1)}$ for the first boundary value problems for a solid cylinder (inner problem) and a cylindrical rigid inclusion (outer problem).

The differential equation describing $\tilde{\boldsymbol{u}}_n^{(-1)}$ of the first boundary value problem in a 3-D domain V is given by (e.g. Markenscoff and Paukshto, 1998)

$$\nabla \times \tilde{\boldsymbol{u}}_n^{(-1)} = \mathbf{0} \quad \text{in } V \tag{1a}$$

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$$\tilde{\boldsymbol{u}}_n^{(-1)} = \mathbf{0} \quad \text{on } \partial V \tag{1b}$$

where ∂V is the surface of V.

We will now restrict attention to 2-D domains and use the cylindrical coordinate system (r, θ). $\tilde{\boldsymbol{u}}_n^{(-1)}$ now takes the following form

$$\tilde{\boldsymbol{u}}_n^{(-1)}(r, \theta) = u_{nr}(r, \theta)\boldsymbol{e}_r + u_{n\theta}(r, \theta)\boldsymbol{e}_\theta \tag{2}$$

By substituting Eq. (2) into Eq. (1), we obtain a general solution of $\tilde{\boldsymbol{u}}_n^{(-1)}$ as follows

$$u_{nr}(r, \theta) = f_n(r, \theta) \tag{3a}$$

$$u_{n\theta}(r, \theta) = \frac{1}{r}\int \frac{\partial f_n(\tau, \theta)}{\partial \theta}\, \mathrm{d}\tau \tag{3b}$$

where $f_n(r, \theta)$ is an arbitrary function vanishing on $\partial\Omega$. To separate the variables r and θ, let

$$f_n(r, \theta) = R_n(r)Q_n(\theta) \tag{4}$$

where $R_n(r)$ and $Q_n(\theta)$ are functions to be determined. Equation (3) is now written as

$$u_{nr}(r, \theta) = R_n(r)Q_n(\theta) \tag{5a}$$

$$u_{n\theta}(r, \theta) = \frac{1}{r}\int R_n(\tau)\, \mathrm{d}\tau \frac{\mathrm{d}Q_n(\theta)}{\mathrm{d}\theta} \tag{5b}$$

from Eq. (5) the divergence of $\tilde{\boldsymbol{u}}_n^{(-1)}$ assumes

$$\nabla \cdot \tilde{\boldsymbol{u}}_n^{(-1)}(r, \theta) = \left(\frac{R_n}{r} + \frac{\mathrm{d}R_n}{\mathrm{d}r}\right)Q_n + \frac{1}{r^2}\int R_n(\tau)\, \mathrm{d}\tau \frac{\mathrm{d}^2 Q_n}{\mathrm{d}\theta^2} \tag{6}$$

To let the Cosserat eigenvectors $\tilde{\boldsymbol{u}}_n^{(-1)}$ be orthogonal and complete in θ, $Q_n(\theta)$ naturally must be of the form of $\cos n\theta$ and $\sin n\theta$. The solutions of the Cosserat eigenvector $\tilde{\boldsymbol{u}}_n^{(-1)}$ Eq. (5) and its divergence Eq. (6) are now taking the form

$$u_{nr}(r, \theta) = R_n(r)\begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix} \tag{7a}$$

$$u_{n\theta}(r, \theta) = \frac{1}{r}\int R_n(\tau)\, \mathrm{d}\tau \begin{Bmatrix} -n \sin n\theta \\ n \cos n\theta \end{Bmatrix} \tag{7b}$$

$$\nabla \cdot \tilde{\boldsymbol{u}}_n^{(-1)}(r, \theta) = \left[\frac{R_n}{r} + \frac{\mathrm{d}R_n}{\mathrm{d}r} - \frac{n^2}{r^2}\int R_n(\tau)\, \mathrm{d}\tau\right]\begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix} \tag{8}$$

where $n = 0, 1, 2, \ldots$.

2. A solid cylinder (inner problem)

In general, eigenvectors corresponding to the same eigenvalue can be made orthogonal. We now discuss the orthogonalization process of the Cosserat eigenvector $\tilde{\boldsymbol{u}}_n^{(-1)}$.

Because of the orthogonality property of $\begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix}$, the orthogonality condition (Mikhlin, 1973) for the first boundary value problem will be automatically satisfied if we choose different values for the parameter n in Eq. (8). However, in case of the same value of n, by properly choosing the function $R_n(r)$, we can also retain the orthogonality condition for the Cosserat eigenvector $\tilde{\boldsymbol{u}}_n^{(-1)}$, a new subscript index p is introduced to distinguish the difference in $R_n(r)$.

For a solid cylinder $r \leqslant r_0$, where r_0 is the radius of the cylinder, the Cosserat eigenvector $\tilde{\boldsymbol{u}}_n^{(-1)}$ and its divergence, together with the boundary condition, are now written in a more general form

$$\tilde{\boldsymbol{u}}_{np}^{(-1)}(r, \theta) = u_{npr}(r, \theta)\boldsymbol{e}_r + u_{np\theta}(r, \theta)\boldsymbol{e}_\theta \tag{9a}$$

$$u_{npr}(r, \theta) = R_{np}(r)\begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix} \tag{9b}$$

$$u_{np\theta}(r, \theta) = \frac{1}{r}\int_{r_0}^{r} R_{np}(\tau)\,\mathrm{d}\tau \begin{Bmatrix} -n \sin n\theta \\ n\ \cos n\theta \end{Bmatrix} \tag{9c}$$

$$R_{np}(r_0) = 0 \tag{9d}$$

$$\nabla \cdot \tilde{\boldsymbol{u}}_n^{(-1)}(r, \theta) = C_{np} f_{np}(r)\begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix} \tag{10a}$$

$$\frac{R_{np}}{r} + \frac{\mathrm{d}R_{np}}{\mathrm{d}r} - \frac{n^2}{r^2}\int_{r_0}^{r} R_{np}(\tau)\,\mathrm{d}\tau = C_{np} f_{np}(r) \tag{10b}$$

where the constant C_{np} and function $f_{np}(r)$ will be determined by the orthogonality condition for the first boundary value problem and $n = 0, 1, 2, \ldots$.

Firstly, we need to make $\tilde{\boldsymbol{u}}_{np}^{(-1)}$ orthogonal to the discrete eigenvector $\tilde{\boldsymbol{u}}_n$. By introducing a potential function

$$\phi_{np}(r) - \phi_{np}(r_0) = \int_{r_0}^{r} R_{np}(\tau)\,\mathrm{d}\tau \tag{11}$$

Eq. (10b) becomes

$$\frac{\mathrm{d}^2\phi_{np}(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\phi_{np}(r)}{\mathrm{d}r} - \frac{n^2}{r^2}\left[\phi_{np}(r) - \phi_{np}(r_0)\right] = C_{np} f_{np}(r) \tag{12a}$$

with the boundary conditions

$$\frac{\mathrm{d}\phi_{np}}{\mathrm{d}r} = 0 \quad r = r_0 \tag{12b}$$

By using the Green's function method, the solution to Eq. (12) for $n > 0$ is given by

$$\phi_{np}(r) - \phi_{np}(r_0) = -\frac{C_{np}}{2n} r^n \int_r^{r_0} \left(r_0^{-2n}\tau^{n+1} + \tau^{-n+1}\right) f_{np}(\tau)\,\mathrm{d}\tau - \frac{C_{np}}{2n}\left(r_0^{-2n}r^n + r^{-n}\right)\int_0^r \tau^{n+1} f_{np}(\tau)\,\mathrm{d}\tau \quad n > 0 \tag{13}$$

Directly integrating Eq. (10b) for $n = 0$ and differentiating Eq. (13) give $R_{np}(r)$ as follows

$$R_{0p}(r) = \frac{C_{0p}}{r}\int_0^r \tau f_{0p}(\tau)\,\mathrm{d}\tau \tag{14a}$$

$$R_{np}(r) = -\frac{C_{np}}{2} r^{n-1}\int_r^{r_0}\left(r_0^{-2n}\tau^{n+1} + \tau^{-n+1}\right) f_{np}(\tau)\,\mathrm{d}\tau - \frac{C_{np}}{2}\left(r_0^{-2n}r^{n-1} - r^{-(n+1)}\right)\int_0^r \tau^{n+1} f_{np}(\tau)\,\mathrm{d}\tau \quad n > 0 \tag{14b}$$

Applying the condition at $r = r_0$ on Eqs. (13) and (14a) shows that $f_{np}(r)$ should satisfy

$$\int_0^{r_0} r^{n+1} f_{np}(r)\,\mathrm{d}r = 0 \quad n = 0, 1, 2, \ldots \tag{15}$$

Equation (15) also serves the orthogonality condition between the Cosserat eigenvectors $\tilde{\boldsymbol{u}}_{np}^{(-1)}$ and $\tilde{\boldsymbol{u}}_n$ for $n \geqslant 1$.

Secondly, we need to make $\tilde{\boldsymbol{u}}_{np}^{(-1)}$ orthonormal inside the subspace $\tilde{\boldsymbol{u}}^{(-1)}$ by (Mikhlin, 1973)

$$\int_A \nabla\cdot\tilde{\boldsymbol{u}}_{np}^{(-1)}(r, \theta)\nabla\cdot\tilde{\boldsymbol{u}}_{mq}^{(-1)}(r, \theta)\,\mathrm{d}A = \delta_{nm}\delta_{pq} \tag{16}$$

where $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$, $r \in [0, r_0]$, $\theta \in [0, 2\pi]$, is the area element in the cylindrical coordinate system (r, θ).

To construct the orthogonal basis for $\tilde{\boldsymbol{u}}_{np}^{(-1)}$, we introduce a new variable

$$s = \frac{r}{r_0} \tag{17}$$

In the coordinate system (s, θ), we rewrite the Cosserat eigenvector $\tilde{\boldsymbol{u}}_{np}^{(-1)}$ Eq. (9) and its divergence Eq. (10) as follows

$$\tilde{\boldsymbol{u}}_{np}^{(-1)}(s, \theta) = u_{npr}(s, \theta)\boldsymbol{e}_r + u_{np\theta}(s, \theta)\boldsymbol{e}_\theta \tag{18a}$$

$$u_{npr}(s, \theta) = R_{np}(s)\begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix} \tag{18b}$$

$$u_{np\theta}(s, \theta) = \frac{1}{s}\int_1^s R_{np}(t)\,\mathrm{d}t \begin{Bmatrix} -n\sin n\theta \\ n\cos n\theta \end{Bmatrix} \tag{18c}$$

$$R_{0p}(s) = \frac{C_{0p} r_0}{s}\int_{t=0}^s t f_{0p}(t)\,\mathrm{d}t \tag{18d}$$

$$R_{np}(s) = -\frac{C_{np}r_0}{2}s^{n-1}\int_{t=s}^{1}\left(t^{n+1}+t^{-n+1}\right)f_{np}(t)\,\mathrm{d}t - \frac{C_{np}r_0}{2}\left(s^{n-1}-s^{-(n+1)}\right)\int_{t=0}^{s}t^{n+1}f_{np}(t)\,\mathrm{d}t \quad n>0 \tag{18e}$$

$$\nabla\cdot\tilde{\boldsymbol{u}}_{np}^{(-1)}(s,\theta) = C_{np}f_{np}(s)\begin{Bmatrix}\cos n\theta\\ \sin n\theta\end{Bmatrix} \tag{19a}$$

$$\frac{R_{np}}{s}+\frac{\mathrm{d}R_{np}}{\mathrm{d}s}-\frac{n^2}{s^2}\int_1^s R_{np}(t)\,\mathrm{d}t = C_{np}r_0f_{np}(s) \tag{19b}$$

The existence condition Eq. (15) for $\tilde{\boldsymbol{u}}_{np}^{(-1)}$ now takes the form

$$\int_0^1 s^{n+1}f_{np}(s)\,\mathrm{d}s = 0 \quad n=0,1,2,\ldots \tag{20}$$

The orthonormality condition Eq. (16) becomes

$$\int_A \nabla\cdot\tilde{\boldsymbol{u}}_{np}^{(-1)}(s,\theta)\nabla\cdot\tilde{\boldsymbol{u}}_{mq}^{(-1)}(s,\theta)\,\mathrm{d}A = \delta_{nm}\delta_{pq} \tag{21}$$

where $\mathrm{d}A = r_0^2 s\,\mathrm{d}s\,\mathrm{d}\theta$, $s\in[0,1]$, $\theta\in[0,2\pi]$, is the area element in the cylindrical coordinate system (s,θ).

Substituting Eq. (19a) into Eq. (21) with $n=m$, we have

$$\pi r_0^2C_{np}C_{nq}\int_0^1 f_{np}(s)f_{nq}(s)s\,\mathrm{d}s = \delta_{pq} \tag{22}$$

Rewrite Eq. (20) and Eq. (22) in the form

$$\int_0^1 1\cdot\frac{f_{np}(s)}{s^n}s^{2n+1}\,\mathrm{d}s = 0 \quad n=0,1,2,\ldots \tag{23}$$

$$\pi r_0^2C_{np}C_{nq}\int_0^1\frac{f_{np}(s)}{s^n}\frac{f_{nq}(s)}{s^n}s^{2n+1}\,\mathrm{d}s = \delta_{pq} \tag{24}$$

Equation (24) shows that the functions $f_{np}(s)/s^n$ have to be chosen as some orthogonal functions with weight $w(s)=s^{2n+1}$. Equation (23) implies that these functions have to start with the term $f_{n0}(s)/s^n=1$ and this term should be excluded in the subspace for $\tilde{\boldsymbol{u}}_{np}^{(-1)}$. The Jacobi polynomials with different weight functions satisfy these requirements.

As examples, we now construct the first two orthogonal bases $\tilde{\boldsymbol{u}}_{np}^{(-1)}$ with $n=0,1$. The other orthogonal bases for $n>1$ can be constructed in a similar way.

2.1. The zeroth orthogonal basis $\tilde{\boldsymbol{u}}_{0p}^{(-1)}$

We start construction of the orthogonal basis with $n=0$. The divergence of the Cosserat eigenvector $\tilde{\boldsymbol{u}}_{0p}^{(-1)}$ is given by

$$\nabla\cdot\tilde{\boldsymbol{u}}_{0p}^{(-1)} = C_{0p}f_{0p}(s) \tag{25}$$

The orthonormality condition Eqs. (23) and (24) take the form

$$\int_0^1 1\cdot f_{0p}(s)s\,\mathrm{d}s = 0 \quad p = 1, 2, 3\ldots \tag{26}$$

$$\pi r_0^2 C_{0p} C_{0_q}\int_0^1 f_{0_p}(s) f_{0_q}(s)s\,\mathrm{d}s = \delta_{pq} \quad p, q = 1, 2, 3, \ldots \tag{27}$$

Equation (27) suggests that we choose

$$f_{0p}(s) = J(p, 1, s) \quad p = 1, 2, 3, \ldots \tag{28}$$

where $J(p, 1, s)$ is the Jacobi polynomial of degree p with weight $w(s) = s$. The Jacobi polynomial $J(p, m, s)$ of degree p with weight $w(s) = s^m$ and its norm $h(p, m)$ are given by (Abramowitz and Stegun, 1972)

$$\int_0^1 J(p, m, s) J(q, m, s) s^m\,\mathrm{d}s = h(p, m)\delta_{pq} \tag{29a}$$

$$J(p, m, s) = \frac{\Gamma(p+m+1)}{\Gamma(2p+m+1)}\sum_{l=0}^{p}(-1)^l\binom{p}{l}\frac{\Gamma(2p+m+1-l)}{\Gamma(p+m+1-l)}s^{p-1} \tag{29b}$$

$$h(p, m) = \frac{\Gamma^2(p+1)\Gamma^2(p+m+1)}{(2p+m+1)\Gamma^2(2p+m+1)} \tag{29c}$$

where $\Gamma(p)$ is the Gamma function and $\binom{p}{l} = \frac{p!}{l!(p-1)!}$ is the binomial coefficient.

We now have the Cosserat eigenvector $\tilde{\boldsymbol{u}}_{0p}^{(l-1)}$ and its divergence as follows

$$\tilde{\boldsymbol{u}}_{0p}^{(-1)} = u_{0_{pr}}\boldsymbol{e}_r + u_{0_{p\theta}}\boldsymbol{e}_\theta \tag{30a}$$

$$u_{0pr} = R_{0p}(s) \tag{30b}$$

$$u_{0p\theta} = 0 \tag{30c}$$

$$R_{0p}(s) = \frac{C_{0p} r_0}{s}\int_{t=0}^{s} tJ(p, 1, t)\,\mathrm{d}t \tag{30d}$$

$$\nabla\cdot\tilde{\boldsymbol{u}}_{0p}^{(-1)} = C_{0p} J(p, 1, s) \tag{31a}$$

$$C_{0p}^2 = \frac{1}{\pi r_0^2 h(p, 1)} \tag{31b}$$

where $p = 1, 2, 3, \ldots$

2.2. The first orthogonal basis $\tilde{\boldsymbol{u}}_{1p}^{(-1)}$

When $n = 1$, the divergence of the Cosserat eigenvector $\tilde{\boldsymbol{u}}_{1p}^{(-1)}$ is given by

$$\nabla \cdot \tilde{\boldsymbol{u}}_{1p}^{(-1)} = C_{1p} f_{1p}(s) \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} \tag{32}$$

The orthonormality conditions Eqs. (23) and (24) take the form

$$\int_0^1 1 \cdot \frac{f_{1p}(s)}{s} s^3 \mathrm{d}s = 0 \quad p = 1, 2, 3, \ldots \tag{33}$$

$$\pi r_0^2 C_{1p} C_{1q} \int_0^1 \frac{f_{1p}(s)}{s} \frac{f_{1q}(s)}{s} s^3 \mathrm{d}s = \delta_{pq} \quad p, q = 1, 2, 3, \ldots \tag{34}$$

Equation (34) suggests that we choose

$$\frac{f_{1p}(s)}{s} = J(p, 3, s) \quad p = 1, 2, 3, \ldots \tag{35}$$

where $J(p, 3, s)$ is the Jacobian polynomial of degree p with weight $w(s) = s^3$.

We now have the Cosserat eigenvector $\tilde{\boldsymbol{u}}_{1p}^{(-1)}$ and its divergence as follows

$$\tilde{\boldsymbol{u}}_{1p}^{(-1)} = u_{1pr} \boldsymbol{e}_r + u_{1_{p\theta}} \boldsymbol{e}_\theta \tag{36a}$$

$$u_{1pr} = R_{1p}(s) \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} \tag{36b}$$

$$u_{1p\theta} = \frac{1}{s} \int_1^s R_{1p}(t)\, \mathrm{d}t \begin{Bmatrix} -\sin\theta \\ \cos\theta \end{Bmatrix} \tag{36c}$$

$$R_{1p}(s) = -\frac{C_{1p} r_0}{2} \int_{t=s}^1 \left(t^3 + t\right) J(p, 3, t)\, \mathrm{d}t - \frac{C_{1p} r_0}{2} (1 - s^{-2}) \int_{t=0}^s t^3 J(p, 3, t)\, \mathrm{d}t \tag{36d}$$

$$\nabla \cdot \tilde{\boldsymbol{u}}_{1p}^{(-1)} = C_{1p} s J(p, 3, s) \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} \tag{37a}$$

$$C_{1p}^2 = \frac{1}{\pi r_0^2 h(p, 3)} \tag{37b}$$

where $p = 1, 2, 3, \ldots$.

3. Cylindrical rigid inclusion (outer problem)

For a cylindrical rigid inclusion $r \geqslant r_0$, where r_0 is the radius of the inclusion, $\tilde{\boldsymbol{u}}^{(-1)}$ and its divergence are derived in a similar manner and given as follows

$$\tilde{\boldsymbol{u}}_{np}^{(-1)}(r,\theta) = u_{npr}(r,\theta)\boldsymbol{e}_r + u_{np_\theta}(r,\theta)\boldsymbol{e}_\theta \tag{38a}$$

$$u_{npr}(r,\theta) = R_{np}(r)\begin{Bmatrix}\cos n\theta\\ \sin n\theta\end{Bmatrix} \tag{38b}$$

$$u_{np\theta}(r,\theta) = \frac{1}{r}\int_{r_0}^{r} R_{np}(\tau)\,\mathrm{d}\tau\begin{Bmatrix}-n\sin n\theta\\ n\cos n\theta\end{Bmatrix} \tag{38c}$$

$$R_{np}(r_0) = 0, \quad R_{np}(\infty) = 0 \tag{38d}$$

$$\nabla\cdot\tilde{\boldsymbol{u}}_{np}^{(-1)}(r,\theta) = C_{np}f_{np}(r)\begin{Bmatrix}\cos n\theta\\ \sin n\theta\end{Bmatrix} \tag{39a}$$

$$\frac{R_{np}}{r} + \frac{\mathrm{d}R_{np}}{\mathrm{d}r} - \frac{n^2}{r^2}\int_{r_0}^{r} R_{np}(\tau)\,\mathrm{d}\tau = C_{np}f_{np}(r) \tag{39b}$$

where the constant C_{np} and function $f_{np}(r)$ will be determined by the orthogonality condition for the first boundary value problem and $n = 0, 1, 2, \ldots$.

To construct the orthogonal basis $\tilde{\boldsymbol{u}}_{np}^{(-1)}$, first, we need to make $\tilde{\boldsymbol{u}}_{np}^{(-1)}$ orthogonal to the discrete eigenvector $\tilde{\boldsymbol{u}}_n$. By introducing a potential function

$$\phi_{np}(r) - \phi_{np}(r_0) = \int_{r_0}^{r} R_{np}(\tau)\,\mathrm{d}\tau \tag{40}$$

Equation (39b) becomes

$$\frac{\mathrm{d}^2\phi_{np}(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\phi_{np}(r)}{\mathrm{d}r} - \frac{n^2}{r^2}\left[\phi_{np}(r) - \phi_{np}(r_0)\right] = C_{np}f_{np}(r) \tag{41a}$$

with the boundary conditions

$$\frac{\mathrm{d}\phi_{np}}{\mathrm{d}r} = 0 \quad r = r_0 \quad \text{and} \quad r = \infty \tag{41b}$$

By using the Green's function method, the solution to Eq. (41) for $n > 0$ is given by

$$\phi_{np}(r) - \phi_{np}(r_0) = -\frac{C_{np}}{2n}r^{-n}\int_{r_0}^{r}\left(\tau^{n+1} + r_0^{2n}\tau^{-n+1}\right)f_{np}(\tau)\,\mathrm{d}\tau$$

$$-\frac{C_{np}}{2n}\left(r^n + r_0^{2n}r^{-n}\right)\int_{r}^{\infty}\tau^{-n+1}f_{np}(\tau)\,\mathrm{d}\tau \quad n > 0 \tag{42}$$

Directly integrating Eq. (39b) for $n = 0$ and differentiating Eq. (42) gives $R_{np}(r)$ as follows

$$R_{0p}(r) = \frac{C_{0p}}{r}\int_{r_0}^{r}\tau f_{0p}(\tau)\,\mathrm{d}\tau \tag{43a}$$

$$R_{np}(r)=\frac{C_{np}}{2}r^{-(n+1)}\int_{r_0}^{r}\left(\tau^{n+1}+r_0^{2n}\tau^{-n+1}\right)f_{np}(\tau)\,\mathrm{d}\tau$$

$$-\frac{C_{np}}{2}\left(r^{n-1}-r_0^{2n}r^{-(n+1)}\right)\int_{r}^{\infty}\tau^{-n+1}f_{np}(\tau)\,\mathrm{d}\tau\quad n>0 \tag{43b}$$

From Eq. (42), we see that the solution $\phi_{np}(r)$ exists if and only if

$$\int_{r_0}^{\infty}r^{-n+1}f_{np}(r)\,\mathrm{d}r=0\quad n=1,2,\ldots \tag{44}$$

Equation (44) also serves the orthogonality condition between $\tilde{\boldsymbol{u}}_{np}^{(-1)}$ and $\tilde{\boldsymbol{u}}_n$ for $n\geqslant 2$. Note that there are no discrete Cosserat eigenvectors corresponding to $n=0$ and $n=1$.

Secondly, we need to make $\tilde{\boldsymbol{u}}_{np}^{(-1)}$ orthonormal inside the subspace $\tilde{\boldsymbol{u}}_{np}^{(-1)}$ by (Mikhlin, 1973)

$$\int_A\nabla\cdot\tilde{\boldsymbol{u}}_{np}^{(-1)}(r,\theta)\nabla\cdot\tilde{\boldsymbol{u}}_{mq}^{(-1)}(r,\theta)\,\mathrm{d}A=\delta_{nm}\delta_{pq} \tag{45}$$

where $\mathrm{d}A=r\,\mathrm{d}r\,\mathrm{d}\theta$, $r\in[r_0,\infty]$, $\theta\in[0,2\pi]$.

To construct the orthogonal basis for $\tilde{\boldsymbol{u}}_{np}^{(-1)}$, we introduce a new variable

$$s=\frac{r_0}{r} \tag{46}$$

In the coordinate system (s,θ), we rewrite the Cosserat eigenvector $\tilde{\boldsymbol{u}}_{np}^{(-1)}$ Eq. (38) and its divergence Eq. (39) as follows

$$\tilde{\boldsymbol{u}}_{np}^{(-1)}(s,\theta)=u_{npr}(s,\theta)\boldsymbol{e}_r+u_{np\theta}(s,\theta)\boldsymbol{e}_\theta \tag{47a}$$

$$u_{npr}(s,\theta)=R_{np}(s)\begin{Bmatrix}\cos n\theta\\ \sin n\theta\end{Bmatrix} \tag{47b}$$

$$u_{np\theta}(s,\theta)=-s\int_1^s R_{np}(t)\frac{\mathrm{d}t}{t^2}\begin{Bmatrix}-n\sin n\theta\\ n\cos n\theta\end{Bmatrix} \tag{47c}$$

$$R_{0p}(s)=-C_{0p}r_0s\int_{t=1}^{s}t^{-3}f_{0p}(t)\,\mathrm{d}t$$

$$R_{np}(s)=-\frac{C_{np}r_0}{2}s^{n+1}\int_{t=1}^{s}\left(\frac{1}{t^{n+3}}+t^{n-3}\right)f_{np}(t)\,\mathrm{d}t-\frac{C_{np}r_0}{2}\left(s^{-n+1}-s^{n+1}\right)\int_{t=0}^{s}t^{n-3}f_{np}(t)\,\mathrm{d}t\quad n>0 \tag{47d}$$

$$\nabla\cdot\tilde{\boldsymbol{u}}_{np}^{(-1)}(s,\theta)=C_{np}f_{np}(s)\begin{Bmatrix}\cos n\theta\\ \sin n\theta\end{Bmatrix} \tag{48a}$$

$$\frac{s^2}{r_0}\left[\frac{R_{np}}{s}-\frac{\mathrm{d}R_{np}}{\mathrm{d}s}+n^2\int_1^s R_{np}(t)\frac{\mathrm{d}t}{t^2}\right]=C_{np}f_{np}(s) \tag{48b}$$

The orthogonal conditions Eqs. (44) and (45) becomes

$$\int_0^1 1\cdot\frac{f_{np}(s)}{s^n}s^{2n-3}\mathrm{d}s=0\quad n=2,\,3,\ldots \tag{49}$$

$$\pi r_0^2 C_{np}C_{nq}\int_0^1\frac{f_{np}(s)}{s^n}\frac{f_{nq}(s)}{s^n}s^{2n-3}\,\mathrm{d}s=\delta_{pq} \tag{50}$$

When $n\geqslant 2$, Eq. (50) shows that the functions $f_{np}(s)/s^n$ have to be chosen as some orthogonal functions with weight $w(s)=s^{2n-3}$. Equation (49) implies that these functions have to start with $f_{n0}(s)/s^n=1$ and this term should be excluded in the subspace for $\tilde{\boldsymbol{u}}_{np}^{(-1)}$. The Jacobi polynomials with different weight functions satisfy these requirements.

In the following sections, we will construct the first three orthogonal bases $\tilde{\boldsymbol{u}}_{np}^{(-1)}$ with $n=0,1,2$. The other orthogonal bases for $n>2$ can be constructed in a similar manner.

3.1. The zeroth orthogonal basis $\tilde{\boldsymbol{u}}_{0p}^{(-1)}$

We start construction of the orthogonal basis with $n=0$. The orthonormality condition Eq. (50) takes the form

$$\pi r_0^2 C_{0p}C_{0q}\int_0^1\frac{f_{0p}(s)}{s^2}\frac{f_{0q}(s)}{s^2}s\,\mathrm{d}s=\delta_{pq}\quad p,\,q=0,\,1,\,2,\ldots \tag{51}$$

Equation (51) suggests that we choose

$$\frac{f_{0p}(s)}{s^2}=J(p,\,1,\,s)\quad p=0,\,1,\,2,\ldots \tag{52}$$

where $J(p, 1, s)$ is the Jacobi polynomial of degree p with weight $w(s)=s$.

We now have the Cosserat eigenvector $\tilde{\boldsymbol{u}}_{0p}^{(-1)}$ and its divergence as follows

$$\tilde{\boldsymbol{u}}_{0p}^{(-1)}=u_{0pr}\boldsymbol{e}_r+u_{0p\theta}\boldsymbol{e}_\theta \tag{53a}$$

$$u_{0pr}=R_{0p}(s) \tag{53b}$$

$$u_{0p\theta}=0 \tag{53c}$$

$$R_{0p}(s)=-C_{0p}r_0 s\int_{t=1}^{s}t^{-1}J(p,\,1,\,t)\,\mathrm{d}t \tag{53d}$$

$$\nabla\cdot\tilde{\boldsymbol{u}}_{0p}^{(-1)}=C_{0p}s^2J(p,\,1,\,s) \tag{54a}$$

$$C_{0p}^2=\frac{1}{\pi r_0^2 h(p,\,1)} \tag{54b}$$

where $p=0,1,2,\ldots$.

3.2. The first orthogonal basis $\tilde{\boldsymbol{u}}_{1p}^{(-1)}$

When $n=1$, the orthonormality conditions Eq. (5) take the form

$$\pi r_0^2 C_{1p} C_{1q} \int_0^1 \frac{f_{1p}(s)}{s^2} \frac{f_{1_q}(s)}{s^2} s \, \mathrm{d}s = \delta_{pq} \quad p, q = 0, 1, 2, \ldots \tag{55}$$

Equation (55) suggests that we choose

$$\frac{f_{1p}(s)}{s^2} = J(p, 1, s) \quad p = 0, 1, 2, \ldots \tag{56}$$

where $J(p, 1, s)$ is the Jacobi polynomial of degree p with weight $w(s)=s$.

We now have the Cosserat eigenvector $\tilde{\boldsymbol{u}}_{1p}^{(-1)}$ and its divergence as follows

$$\tilde{\boldsymbol{u}}_{1p}^{(-1)} = u_{1pr}\boldsymbol{e}_r + u_{1p\theta}\boldsymbol{e}_\theta \tag{57a}$$

$$u_{1pr} = R_{1p}(s) \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} \tag{57b}$$

$$u_{1p\theta} = -s \int_1^s R_{1p}(t) \frac{\mathrm{d}t}{t^2} \begin{Bmatrix} -\sin\theta \\ \cos\theta \end{Bmatrix} \tag{57c}$$

$$R_{1p}(s) = -\frac{1}{2} C_{1p} r_0 s^2 \int_{t=1}^{s} \left(t^{-2}+1\right) J(p, 1, t) \, \mathrm{d}t - \frac{1}{2} C_{1p} r_0 \left(1-s^2\right) \int_{t=0}^{s} J(p, 1, t) \, \mathrm{d}t \tag{57d}$$

$$\nabla \cdot \tilde{\boldsymbol{u}}_{1p}^{(-1)} = C_{1p} s^2 J(p, 1, s) \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} \tag{58a}$$

$$C_{1p}^2 = \frac{1}{\pi r_0^2 h(p, 1)} \tag{58b}$$

where $p = 0, 1, 2, \ldots$.

3.3. The second orthogonal basis $\tilde{\boldsymbol{u}}_{2p}^{(-1)}$

When $n=2$, the orthonormality conditions Eqs. (49) and (50) take the form

$$\int_0^1 1 \cdot \frac{f_{2p}(s)}{s^2} s \, \mathrm{d}s = 0 \quad p = 1, 2, 3, \ldots \tag{59}$$

$$\pi r_0^2 C_{2p} C_{2q} \int_0^1 \frac{f_{2p}(s)}{s^2} \frac{f_{2q}(s)}{s^2} s \, \mathrm{d}s = \delta_{pq} \quad p, q = 1, 2, 3, \ldots \tag{60}$$

Equation (60) suggests that we choose

$$\frac{f_{2p}(s)}{s^2} = J(p, 1, s) \quad p = 1, 2, 3, \ldots \tag{61}$$

where $J(p, 1, s)$ is the Jacobi polynomial of degree p with weight $w(s) = s$.

We now have the Cosserat eigenvector $\tilde{\boldsymbol{u}}_{2p}^{(-1)}$ and its divergence as follows

$$\tilde{\boldsymbol{u}}_{2p}^{(-1)} = u_{2pr}\boldsymbol{e}_r + u_{2p\theta}\boldsymbol{e}_\theta \tag{62a}$$

$$u_{2pr} = R_{2p}(s)\begin{Bmatrix} \cos 2\theta \\ \sin 2\theta \end{Bmatrix} \tag{62b}$$

$$u_{2p\theta} = -s\int_1^s R_{2p}(t)\frac{\mathrm{d}t}{t^2}\begin{Bmatrix} -2\sin 2\theta \\ 2\cos 2\theta \end{Bmatrix} \tag{62c}$$

$$R_{2p}(s) = -\frac{1}{2}C_{2p}r_0 s^3\int_{t=1}^s \left(t^{-3}+t\right)J(p, 1, t)\,\mathrm{d}t - \frac{1}{2}C_{2p}r_0\left(s^{-1}-s^3\right)\int_{t=0}^s tJ(p, 1, t)\,\mathrm{d}t \tag{62d}$$

$$\nabla\cdot\tilde{\boldsymbol{u}}_{2p}^{(-1)} = C_{2p}s^2 J(p, 1, s)\begin{Bmatrix} \cos 2\theta \\ \sin 2\theta \end{Bmatrix} \tag{63a}$$

$$C_{2p}^2 = \frac{1}{\pi r_0^2 h(p, 1)} \tag{63b}$$

where $p = 1, 2, 3, \ldots$.

4. Application to thermoelasticity and viscoelasticity

To illustrate the application of the Cosserat eigenvectors $\tilde{\boldsymbol{u}}^{(-1)}$, we present an example of a heat source in an infinite medium containing a cylindrical rigid inclusion. The thermoelastic problem is described by

$$\Delta\boldsymbol{u} + \omega\nabla\nabla\cdot\boldsymbol{u} = (3\omega - 1)\alpha\nabla T \quad r \geqslant r_0 \tag{64a}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad r = r_0 \tag{64b}$$

where $\omega = (\lambda + \mu)/\mu$, λ and μ are the Lamé constants, r_0 the radius of the inclusion, α the thermal expansion coefficient. Suppose that a non-harmonic temperature field generated by a heat source is

$$T = T_0\left(\frac{r_0}{r}\right)^{4-\beta} \quad r \geqslant r_0 \tag{65}$$

where T_0 and β are constants, $0<\beta<1$.

In thermoelasticity, only the discrete Cosserat eigenvectors $\tilde{\boldsymbol{u}}_n$ make a contribution to the displacement caused by a harmonic temperature field, while both $\tilde{\boldsymbol{u}}_n$ and $\tilde{\boldsymbol{u}}_{np}^{(-1)}$ make a contribution to the displacement caused by a heat source (non-harmonic temperature). The representation theorem for a heat source takes the form (Liu et al., 1998)

$$\boldsymbol{u} = (3\omega - 1)\alpha\left[\sum_n \frac{\tilde{\omega}_n}{\tilde{\omega} - 1}(T, \nabla\cdot\tilde{\boldsymbol{u}}_n)\tilde{\boldsymbol{u}}_n + \sum_n\sum_p \frac{1}{1+\omega}\left(T, \nabla\cdot\tilde{\boldsymbol{u}}_{np}^{(-1)}\right)\tilde{\boldsymbol{u}}_{np}^{(-1)}\right] \tag{66}$$

The displacement caused by the axisymmetric temperature field Eq. (65) has a simple form $\boldsymbol{u} = u_r\boldsymbol{e}_r$, which corresponds to the Cosserat eigenvectors with $n = 0$. There are no discrete eigenvectors corresponding to $n = 0$ and only $\tilde{\boldsymbol{u}}_{0p}^{(-1)}$ make a contribution to the displacement field. Consequently, the representation theorem Eq. (66) reduces to

$$\boldsymbol{u} = \frac{(3\omega - 1)\alpha}{\omega + 1}\sum_{p=0}^{\infty}\left(T, \nabla\cdot\tilde{\boldsymbol{u}}_{0p}^{(-1)}\right)\tilde{\boldsymbol{u}}_{0p}^{(-1)} \tag{67}$$

Substituting Eqs. (53) and (54) into Eq. (67), we have the non-dimensionalized displacement

$$\overline{u}_r = \frac{u_r(\omega+1)}{(3\omega-1)\alpha r_0 T_0} = \sum_{p=0}^{\infty}\frac{1}{h(p, 1)}\left[\int_{s=0}^{1} s^{3-\beta}J(p, 1, s)\,\mathrm{d}s\right]\left[s\int_{t=s}^{1} t^{-1}J(p, 1, t)\,\mathrm{d}t\right] \tag{68}$$

For comparison, we solve Eq. (64) subjected to the temperature field Eq. (65) by classical means and obtain the non-dimensionalized displacement component in closed form

$$\overline{u}_r = \frac{u_r(\omega+1)}{(3\omega-1)\alpha r_0 T_0} = \frac{1}{\beta - 2}\left[\left(\frac{r_0}{r}\right)^{3-\beta} - \left(\frac{r_0}{r}\right)\right] \tag{69}$$

The numerical calculations for the approximate solution given by the summation of the first N terms of Eq. (68) and the exact solution given by Eq. (69) are shown in Tables 1 and 2. We see that the summation of the series of $\tilde{\boldsymbol{u}}_{0p}^{(-1)}$ converges very fast to the exact solution. This confirms that the Cosserat spectrum theory is an efficient method for solving problems of general body force and boundary loading. In general, only the first few eigenfunctions are required for a certain geometry and they may be obtained analytically or numerically.

The thermoelastic solution can also be extended to thermoviscoelasticity (Markenscoff et al., 1998). If a material is viscoelastic and the temperature field Eq. (65) is time dependent, the thermoviscoelastic solution in the form of Laplace transform is essentially the same as its thermoelastic counterpart,

Table 1
Numerical result for the displacement $\bar{u}_r\,(r_0/r = 0.5)$

N	$\beta = 0.001$	$\beta = 0.01$	$\beta = 0.1$	$\beta = 0.5$
0	0.173330	0.173721	0.177730	0.198042
1	0.196070	0.196451	0.200340	0.219700
2	0.187550	0.187988	0.192580	0.215270
3	0.187550	0.188004	0.192636	0.215441
4	0.187550	0.188006	0.192649	0.215495
5	0.187550	0.188005	0.192647	0.215485
6	0.187550	0.188005	0.192646	0.215480
7	0.187550	0.188005	0.192646	0.215482
8	0.187550	0.188005	0.192647	0.215483
9	0.187550	0.188005	0.192646	0.215482
10	0.187550	0.188005	0.192646	0.215482
Exact solution	0.187550	0.188005	0.192646	0.215482

Table 2
Numerical result for the displacement $\bar{u}_r\,(r_0/r = 0.2)$

N	$\beta = 0.001$	$\beta = 0.01$	$\beta = 0.1$	$\beta = 0.5$
0	0.160984	0.161347	0.165071	0.183936
1	0.095477	0.095870	0.099938	0.121545
2	0.096043	0.096432	0.100453	0.121839
3	0.096042	0.096419	0.100335	0.121482
4	0.096042	0.096417	0.100317	0.121405
5	0.096042	0.096417	0.100316	0.121399
6	0.096042	0.096417	0.100317	0.121404
7	0.096042	0.096417	0.100317	0.121408
8	0.096042	0.096417	0.100317	0.121408
Exact solution	0.096042	0.096417	0.100317	0.121408

namely,

$$\hat{\boldsymbol{u}} = \frac{(3\hat{\omega} - 1)\alpha}{\hat{\omega} + 1} \sum_{p=0}^{\infty} \left(\hat{T}, \nabla \cdot \tilde{\boldsymbol{u}}_{0p}^{(-1)} \right) \tilde{\boldsymbol{u}}_{0p}^{(-1)} \tag{70}$$

where $\tilde{\omega} = (\hat{\lambda} + \hat{\mu})/\hat{\mu}$, $\hat{\lambda}$ and $\hat{\mu}$ are complex Lamé constants, $\hat{T}$ and $\hat{\boldsymbol{u}}$ are the Laplace transform of T and $\boldsymbol{u}$, respectively. Note that the Cosserat eigenvectors $\tilde{\boldsymbol{u}}_{0p}^{(-1)}$ depend only on the geometry. The inverse Laplace transform will give the displacement field in convolution form for the thermoviscoelastic problem as follows

$$u_r(t) = r_0 \sum_{p=0}^{\infty} \frac{1}{h(p,\,1)} \left[\int_{s=0}^{1} s^{3-\beta} J(p,\,1,\,s)\,\mathrm{d}s \right] \left[s \int_{t=s}^{1} t^{-1} J(p,\,1,\,t)\,\mathrm{d}t \right] \int_0^t T_0(t') G^{(-1)}(t - t')\,\mathrm{d}t' \tag{71a}$$

where $G^{(-1)}(t)$ is the inverse Laplace transform of the moduli given by

$$G^{(-1)}(t) = L^{-1}\left[\frac{(3\hat{\omega} - 1)\alpha}{\hat{\omega} + 1} \right] \tag{71b}$$

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